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DYNAMIC BUCKLING OF AN INELASTIC COLUMN

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Abstract—A quasi-bifurcation theory, for systems subject to configuration-dependent forces, is employed to analyze the dynamic buckling behavior of an inelastic column hinged at both ends. Numerical results are presented and show that only certain eigenmodes of motion predominate at a time, as well as terminally depending on the nature of the time-dependent eigenvalues which, in turn, depend on the geometry, material properties and the loading history of the column.

1. INTRODUCTION

The subject of dynamic buckling of columns under axial loading of short duration has been considered by many investigators. In spite of the numerous studies of the problem, the understanding of the complex phenomenon, which involves the interaction of axial stress waves with flexural bending, is not yet complete, especially that of inelastic columns. The dynamic buckling of a column is influenced by its geometry, initial imperfections or disturbances, material properties, boundary conditions and the loading history. Instability develops from the growth of these imperfections or disturbances and one can assess the lateral displacement profile growth with time for various loading histories. "Buckling" is often defined as that which occurs when the magnitude of lateral deformation exceeds an arbitrarily chosen limit.

In the early analyses of the dynamic buckling of elastic columns[1–6], the axial inertia forces were omitted. This omission is equivalent to assuming that the axial stress wave is propagating with a high velocity such that the axial load is practically uniform along the length during buckling. It has been shown[7] that axial inertia effects are only of secondary significance as far as the overall buckling response of a column is concerned, provided that the rate of loading is not too high. Furthermore, under such conditions, the inclusion of rotary inertia and shear deformations have also relatively little effect on the buckling results[8, 9]. There have been other investigations on the impact stability of elastic columns including axial inertia forces[10, 11] and other simplifying assumptions which may require further analytical and/or experimental verification.

There have been few investigations on the dynamic buckling of inelastic columns. However, a noticeable experimental investigation on the dynamic flexural buckling of rods subject to axial impact has been performed by Abrahamson and Goodier[12, 13]. The observed phenomenon, which involves the interaction between axial non-linear plastic waves and flexural waves is only partially understood[14]. The present paper is concerned with a comparatively simple problem which may illustrate one aspect of the dynamic buckling behavior of inelastic columns where the rates of axial loading are not too high such that axial inertia effects are negligible.

In this paper, a simply supported column made of a bilinear material and subject to arbitrary inertial disturbances is considered. In his statistical analysis, however, Lindberg[15] has found that initial random deflections lead to good agreement with the observed buckling wave lengths of a thin elastic bar subject to axial impact. The lower end of the present column is stationary while the upper end may move downward with a stepwise axial velocity in a short duration. To simplify the analysis, the dynamic Euler–Bernoulli beam theory is employed and the rotary inertia and transverse shear deformation are not considered. The dynamic buckling process is determined by a recent developed quasi-bifurcation theory[16, 17] which is briefly described in the paper. Numerical results for a number of columns are presented and discussed.

It is also the purpose of this paper to employ the present column problem to illustrate that the dynamic stability of a system subject to configuration-dependent forces can be treated as an initial-value-eigenvalue problem which may be solved by the quasi-bifurcation theory.

2. QUASI-BIFURCATION

A certain motion of a system of n degrees of freedom is considered stable if, after a sufficiently small disturbance, the system remains to follow the undisturbed motion. In other words, the undisturbed motion is stable if the deviated motion, $\zeta_r(t)$, $r = 1, \ldots, n$, the difference between the disturbed and undisturbed motions in n generalized coordinates, remains small.

For any deviated motion, stable or not, there is the following identity:

$$\frac{1}{2}\zeta_r(t)\zeta_r(t) = \frac{1}{2}\gamma_r\gamma_r + \int_0^t \left[\dot{\gamma}_r\gamma_r + \int_0^{t'} \ddot{\zeta}_r\zeta_r \,\mathrm{d}\tau + \int_0^{t'} \dot{\zeta}_r\dot{\zeta}_r \,\mathrm{d}\tau\right]\mathrm{d}t' \tag{1}$$

in which dots indicate differentiation with respect to time t and γ_r and $\dot{\gamma}_r$ are the initial values of the deviation motion. When the generalized forces, P_r, of the system are configuration-dependent and when the deviated motion is relatively small, the deviated motion is governed by the following variational equations:

$$\ddot{\zeta}_r = A_{rs}\zeta_s = \Delta P_r, \qquad r = 1, \ldots, n; \quad s = 1, \ldots, n \tag{2}$$

where the coefficients A_{rs} are known functions of time depending on the undisturbed motion only. Equation (1) shows that the undisturbed motion is unstable or the deviation motion is not bounded when the quadratic term

$$Q = \Delta P_r\zeta_r = \ddot{\zeta}_r\zeta_r = A_{rs}\zeta_r\zeta_s \tag{3}$$

is positive after a critical time, t_{cr}, for any possible deviated motion. Furthermore, a variation of Q yields the following relationship:

$$\delta Q = 2A_{rs}\zeta_s\delta\zeta_r = 2\ddot{\zeta}_r\delta\zeta_r \tag{4}$$

which may be employed, by appropriate coordinate transformation, to obtain equations of motion in other generalized coordinates.

The quadratic form Q at a given time can always be reduced to a linear combination of squares [18] such as:

$$Q = \alpha_m\eta_m^{\,2}, \qquad m = 1, \ldots, n \tag{5}$$

where $\alpha_m(t)$ are the time-dependent eignevalues, η_m are related to ζ_r by an orthogonal transformation

$$\zeta_r = l_{mr}\eta_m \tag{6}$$

where l_{mr} are the directional numbers of n mutually orthogonal vectors, η_m, in the ζ_r space known as eigenvectors or as eigenmodes of motion. Such transformation is called a transformation to principal axes. For cases where l_{mr} are always constants, a variation of Q leads to the uncoupled ordinary differential equations

$$\ddot{\eta}_m = \alpha_{\underline{m}}\eta_{\underline{m}}, \qquad m = 1, \ldots, n \tag{7}$$

where the underscores are placed under the indices to suspend the summation convention. If ζ_r and η_r are normalized, Q assumes the greatest value equal to the largest eigenvalue, α_k, corresponding to the configuration η_k in the ζ_r space by a theorem of Weierstrasse [18]. Therefore, α_k and η_k may also be determined by the extremum condition. If $\alpha_k > 0$ and $\alpha_k > \alpha_r$ for $r \neq k$, then the eigenmode of motion, η_k, grows at the highest rate, provided that the initial disturbance has such a component. Such phenomenon is called a quasibifurcation phenomenon. The above concept and approach are applied to the solution of the present problem.

Of an elastic–plastic solid of a volume, V, let the undisturbed motion be described by the displacement $U_K(X_M, t)$ in the X_M spatial coordinate system. The deviated (or additional) displacement is denoted by $u_K(X_M, t)$. The corresponding Lagrangian strains are e_{KL} and ϵ_{KL} and the corresponding Piola–Kirchhoff stresses are S_{KL} and s_{KL}, respectively. When u_k is relatively small, the additional strain, ϵ_{KL}, is given by

$$\epsilon_{KL} = \frac{1}{2}(u_{K,L} + u_{L,K} + U_{M,K}u_{M,L} + U_{M,L}u_{M,K}) \tag{8}$$

and the deviated motion is governed by the following equation of motion:

$$\Delta P_M = [S_{KL}u_{M,L} + s_{KL}(\delta_{ML} + U_{M,L})]_{,K} = \rho \ddot{u}_M \tag{9}$$

where ρ is the initial mass density. The deviated motion satisfies the following boundary conditions: $u_K(X_M, t) = 0$ on that part of the boundary with prescribed kinetic conditions and

$$[S_{KL}u_{M,L} + s_{KL}(\delta_{ML} + U_{M,L})]N_K = 0 \tag{10}$$

on that part of the boundary with prescribed surface forces, where N_K is the outward unit normal to the surface. The quadratic term for the elastic–plastic continuum is given by

$$Q(u_k) = \int_V \Delta P_M u_M \,\mathrm{d}V. \tag{11}$$

For a body which has no interior discontinuities of the variables, the quadratic term may be written as

$$Q(u_K) = -\int_V (S_{KL}u_{M,K}u_{M,L} + s_{KL}\epsilon_{KL})\,\mathrm{d}V. \tag{12}$$

The motion and its stability of the solid may be determined by analyzing the functional $Q(u_K)$.

3. COLUMN MODEL

For convenience of analysis, a long and straight column of a uniform rectangular cross-section and with hinged ends is considered. Its dimensions are length, l, section width, b and depth, h, such that $h < b$ and $h \ll l$. The column has an initial mass density, ρ. The lower end of the column is stationary while the upper end may move vertically downward with an axial velocity, v_0, in the form of a step function. Meanwhile, the column may be subjected to infinitesimal lateral disturbances. The subsequent lateral motion of the column is to be investigated.

It is assumed that the axial velocity is sufficiently slow so that the internal resultant axial force may be considered constant along the entire length of the column. It is also assumed that the effects of rotational inertia and transverse shear are negligible.

For the purpose of this paper, the property of the column material may be characterized as having a bilinear stress–strain relationship which may be described in terms of the following three parameters: Young's modulus, E, initial yield strain, e_0, and a single tangent modulus, E_t ($E_t > 0$, and $E_t < E$) associated with incremental plastic loading. Let the uniform axial stress and strain of the undisturbed motion of the column be denoted by S and e, respectively and let the additional axial stress and strain associated with the lateral motion be denoted by s and ϵ, respectively. The stress–strain relationship may be specifically expressed as:

$$\begin{aligned} \dot{s} &= E_t\dot{\epsilon} \quad \text{for} \quad e + \epsilon = e^* \quad \text{and} \quad (e+\epsilon)(\dot{e}+\dot{\epsilon}) > 0 \\ \dot{s} &= E\dot{\epsilon} \quad \text{for} \quad |e+\epsilon| < |e^*| \quad \text{or} \quad (e+\epsilon)(\dot{e}+\dot{\epsilon}) < 0 \end{aligned} \tag{13}$$

where e^* ($|e^*| > |e_0|$) is the largest local subsequent yielding strain that the material has ever experienced.

4. DYNAMIC BUCKLING

Let the origin of the cartesian axes of reference (x, y, z) be at the centroid of the base, the x-axis being positive upward along the line of centroid and the z-axis being parallel to the depth of the sections. The respective displacement components of the centroid of a section caused by the lateral motion are denoted by u, v and w. For symmetrical bending, $v = 0$. When the strains in the column are relatively small, the uniform axial compressive strain (positive in compression) may be written as:

$$e = \frac{v_0 t}{l} \quad \text{for} \quad t > 0 \tag{14}$$

which increases monotonically with time. Employing the Euler–Bernoulli assumption, the additional axial strain associated with the flexural motion may be expressed by:

$$\epsilon = u_{,x} - zw_{,xx}. \tag{15}$$

The uniform axial stress (positive in compression) is given by:

$$S = E[(1-\beta)e_0 + \beta e] \tag{16}$$

where

$$\beta = \begin{matrix} 1 & \text{for} \quad e < e_0 \\ \dfrac{E_t}{E} & \text{for} \quad e > e_0. \end{matrix} \tag{17}$$

The additional axial stress rate in the early stage of buckling when $\epsilon \ll e$ is given by:

$$\dot{s} = \begin{cases} E\dot{\epsilon} & \text{for} \quad e < e_0 \quad \text{or} \quad \dot{e} + \dot{\epsilon} < 0 \\ E_t\dot{\epsilon} & \text{for} \quad e > e_0 \quad \text{and} \quad \dot{e} + \dot{\epsilon} > 0. \end{cases} \tag{18}$$

Furthermore, when $\dot{\epsilon} < \dot{e}$ in an entire period $0 < t < t^*$, the additional axial stress in that period is given by:

$$s = E\beta\epsilon. \tag{19}$$

The omission of the axial inertia effects is equivalent to the condition that there is no deviation in the axial load during buckling, or

$$b \int_{-h/2}^{h/2} s \, \mathrm{d}z = 0. \tag{20}$$

Substitution of eqns (15) and (19) into (20) yields

$$u_{,x} = 0. \tag{21}$$

Employing eqns (12)–(21) the quadratic functional for the lateral motion of the column in the period $0 < t < t^*$ may be written as:

$$Q = bh \int (Sw_{,x}{}^2 - E\beta k^2 w_{,xx}^2)\, \mathrm{d}x \tag{22}$$

where $k = h/2\sqrt{3}$ is the radius of gyration of the cross-section. The functional Q, may be further simplified by the following transformation:

$$w = \sum_{m=1}^{\infty} \bar{\eta}_m(t) \sin \frac{m\pi x}{l}. \tag{23}$$

Substituting of eqn (23) into eqn (22) yields

$$Q = \frac{bhE}{2b}\sum_{m=1}^{\infty} \alpha_m \bar{\eta}_m(t)^2 \tag{24}$$

where

$$\alpha_m = \beta(m\pi)^2\left[\left(\frac{1}{\beta}-1\right)e_0 + \frac{v_0 t}{l} - \left(\frac{m\pi k}{l}\right)^2\right]. \tag{25}$$

Equations (23) and (24) show that w is expressed in terms of the eigenvectors as given by eqn (23). Employing the variational relationship given by eqn (4) with respect to a specific $\bar{\eta}_m$, the following uncoupled, dimensionless equation of motion is obtained:

$$\frac{\partial^2 \eta_m}{\partial \tau^2} = \alpha_m \alpha_m, \qquad m = 1, 2, 3, \ldots \tag{26}$$

where $\eta_m = \bar{\eta}_m / l$, $\tau = (c_0 t / l)$ and $c_0 = \sqrt{(E/\rho)}$ is the speed of propagation of the axial wave.

Equation (26) and the functional, Q, in eqn (24) show when any one of the α_m becomes positive, that particular η_m mode of motion may grow and the undisturbed axial motion may be unstable. Equation (25) shows that $\alpha_1(t)$ has first the algebraically largest value which changes from negative to positive at the earliest time. In fact, the uniform axial strain corresponding to $\alpha_1 = 0$ is equal to the static buckling strain, e_s, given by

$$e_s = \frac{v_0 t}{l} = \frac{v_0 \tau}{c_0} = \left(\frac{\pi k}{l}\right)^2 - \left(\frac{1}{\beta}-1\right)e_0. \tag{27}$$

Let the time at which $\alpha_m = 0$ be denoted by τ_m or

$$\tau_m = \frac{c_0}{v_0}\left[\left(\frac{m\pi k}{l}\right)^2 - \left(\frac{1}{\beta}-1\right)e_0\right], \qquad m = 1, 2, 3, \ldots. \tag{28}$$

It is noted that the bilinear material property may cause α_m to have a jump in value at $\tau = c_0 e_0 / v_0$. If $\alpha_m < 0$ for $e < e_0$ and $\alpha_m > 0$ for $e > e_0$, then $\alpha_m = 0$ occurs at $\tau_m = c_0 e_0 / v_0$.

In the time interval $\tau_1 < \tau < \tau_2$, $\alpha_1 > 0$ and $\alpha_m < 0$ for $m = 2, 3, 4, \ldots$, only the α_1 mode of motion grows montonically. If the velocity ratio v_0/c_0 is infinitesimal, the time interval $\tau_1 < \tau < \tau_2$ is very long. The α_1 mode of motion grows only in that period and leads to the buckling failure of the column in that particular buckling mode and at an axial strain slightly higher than e_s. If the velocity ratio, v_0/c_0, is comparatively large, τ_1, τ_2, τ_3, etc. may be close together and a number of eigenmodes of motion may be activated at the same time. In a period $\tau_n < \tau < \tau_{n+1}$, all α_m with $m \le n$ are positive except that α_1 may or may not have the largest positive value depending on the geometry, material properties and rate of loading, v_0, of the column. To determine quantitively dynamic buckling behavior of an inelastic column, it is convenient to integrate eqn (26) by a numerical procedure. A computer program based on the fourth order Runge–Kutta method has been prepared for this purpose. A number of cases with various combinations of parameters have been considered and are described as follows.

5. NUMERICAL RESULTS

Numerical results have been obtained by the foregoing procedure for a number of columns having the same material properties: $E_t/E = 0.1$ and $e_0 = 0.005$. All the columns are subjected to initial disturbances producing an initial displacement profile which has the components of all eigenmodes of deformation of an equal amplitude such that $\eta_m(0) = +0.00001$ or -0.00001 (but not simultaneously) and $\dot{\eta}_m(0) = 0$ for $m = 1, 2, 3, \ldots$. The typical results are shown in Figs. 1–6.

Figure 1 shows the histories of the amplitudes of a number of eigenmodes of motion of Case 1 having the parameters: $v_0/c_0 = 0.0001$ and $(\pi k/l) = 0.002$. The static buckling strain of this

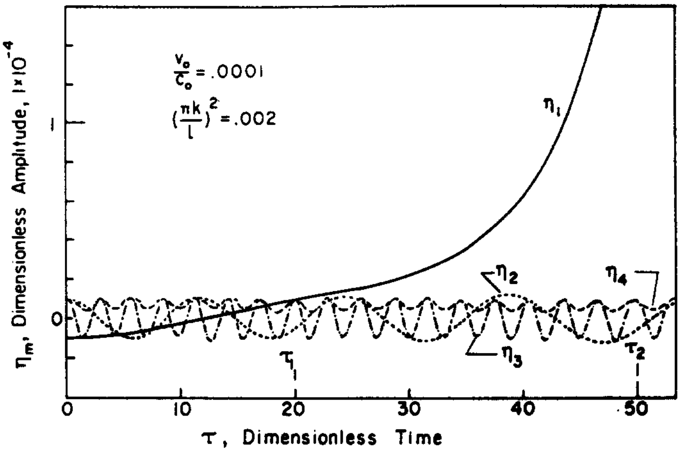

Fig. 1. Eigenmodes of motion of Case 1.

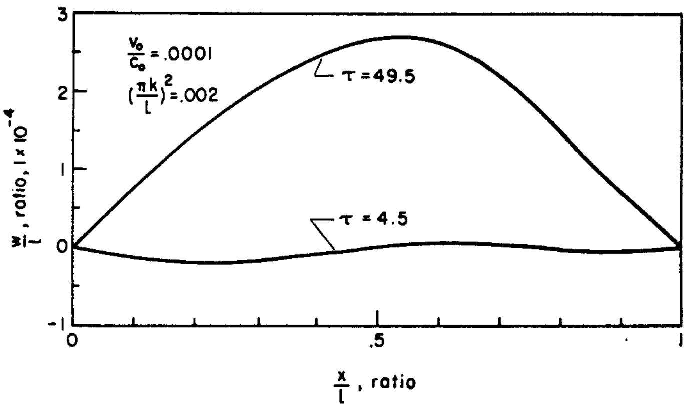

Fig. 2. Deformed profiles of Case 1.

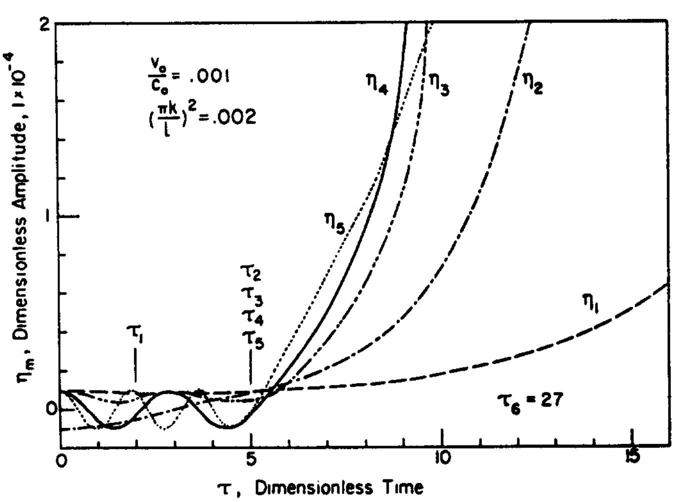

Fig. 3. Eigenmodes of motion of Case 2.

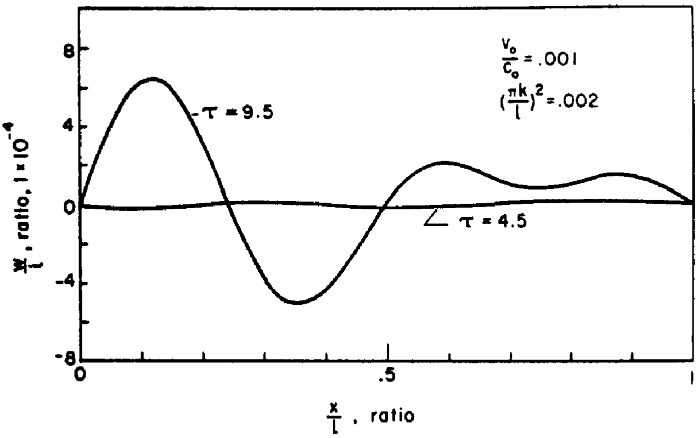

Fig. 4. Deformed profiles of Case 2.

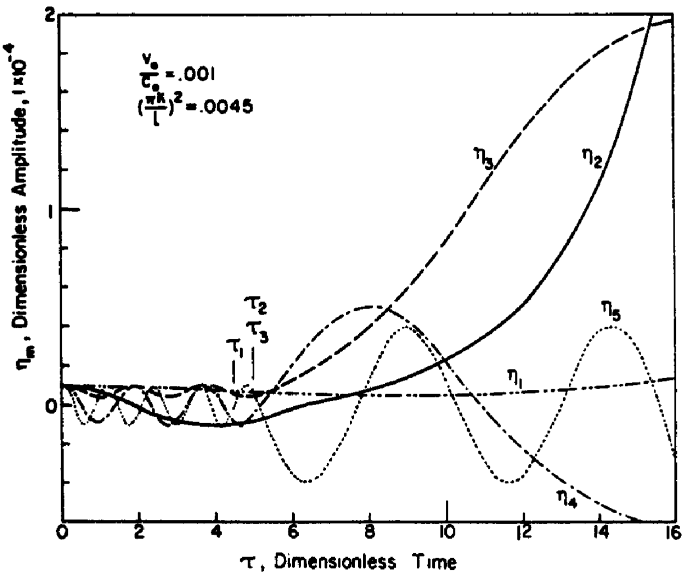

Fig. 5. Eigenmodes of motion of Case 3.

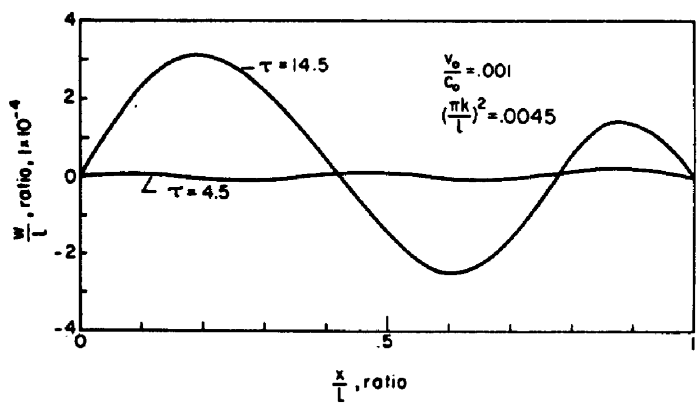

Fig. 6. Deformed profiles of Case 3.

column is in the elastic range and is reached at τ_1. The loading velocity is relatively slow such that the first eigenmode of motion predominates and grows until the collapse of the column before any other mode of motion can grow. The deformed profiles of the column at two different times are shown in Fig. 2. Here, the dynamic buckling mode is similar to the static buckling mode, but the dynamic axial strain at collapse is about 2.5 times larger than the static buckling strain.

Figure 3 shows the histories of the amplitudes of a number of eigenmodes of motion of Case 2 having the parameters $v_0/c_0 = 0.001$ and $(\pi k/l)^2 = 0.002$. This column is identical to that of Case 1 except the loading velocity is ten times higher. Due to the abrupt change in material stiffness and the higher loading velocity, there are four eigenmodes of motion activated at $\tau = 5$ when the initial yield strain is reached. However, out of the four modes, η_4 mode of motion grows at the highest rate, as shown in Fig. 3. The deformed profiles of the column at $\tau = 4.5$ and $\tau = 9.5$ are shown in Fig. 4. Soon after $\tau = 9.5$ and in some regions of the column, the deflections are large. Unloading occurs partially over the cross-sections and the determination of the further motion would require another extended nonlinear analysis which may be considered in the future.

Figures 5 and 6 show the results of Case 3 having the parameters $v_0/c_0 = 0.001$ and $(\pi k/l)^2 = 0.0045$. Of this comparatively stout column, only η_2 and η_3 modes of motion are activated at the initial yield strain at $\tau = 5$, and η_2 mode of motion grows at the highest rate. It is of interest to note that the amplitude of oscillation of η_4 or η_5 mode of motion changes from a small to a relatively larger value, as shown in Fig. 5, when the eigenvalue changes abruptly and when the stiffness of the material is reduced from that of Young's modulus, E, to that of tangent models, E_t, at $\tau = 5$. However, as long as α_4 or α_5 is negative, the η_4 or η_5 mode of motion remains oscillatory and bounded within the period shown.

6. CONCLUDED REMARKS

The results of this paper indicate that the approach presented herein may be employed to determine the dynamic buckling behavior of an elastic–plastic column under certain axial loadings. The perturbed flexural motion can be expressed as a sum of the uncoupled eigenmodes of motion of the column. The eigenmodes of motion can be obtained from a quadratic functional. The eigenmodes of motion depend on the undisturbed motion, geometry, material properties and loading history of the column. The initiation of an eigenmode of motion requires the presence of that particular mode of initial disturbance, no matter how small it is. The development of each eigenmode of motion, however, depends strongly on the nature of the time-dependent eigenvalue. When all the eigenmodes of motion are initially perturbed, only certain eigenmodes of motion predominate at a time depending on the characteristics of the functional, Q, which describes inherently the intrinsic interactions between the undisturbed and disturbed motion of the column.

It is noted that "$\delta Q = 0$" is the quasi-static bifurcation criterion employed by Hill[19]. It is obvious that quasi-static bifurcation phenomena are naturally a sub-class of dynamic quasi-bifurcation phenomena, as partially illustrated in this paper. However, to understanding fully the dynamic quasi-bifurcation phenomena of elastic–plastic continua, further studies are required.

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